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A POWERFUL LL(K) COVERING TRANSFORMATION*

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Abstract. k-transformable grammars have been conjectured to be the uppermost class of LL(k) covering transformable grammars. PLR(k) grammars have been known as a well characterized subclass of k-transformable grammars. Being contrary to those claims, this paper shows that some PLR(k) grammars are not k-transformable, and so k-transformable grammars are not the true uppermost.

A powerful LL(k) covering transformation is suggested in this paper. It is a generalization of the transformations of k-transformable grammars and PLR(k) grammars. A remarkable aspect of the new transforming process is the deterministic property, where "deterministic" means that the transformation is obtained in a single process without requiring any heuristic, unlike k-transformable grammars' transformation for which a heuristic is required. The transformable grammar class is shown to be larger than k-transformable grammars and PLR(k) grammars.

Key words. compilers, LR grammars, LL grammars, left-to-right cover

AMS subject classifications. 68N20, 68Q42, 68Q45

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1. Introduction. LR grammars and LL grammars are two of the most representative context-free grammars for programming languages. When we compare the parsers of those grammars, we see that LL parsers are easier and simpler to implement, understand, and error recover and have a greater flexibility in computing semantic attributes during parsing, but LL grammars are a small subclass of LR grammars. Some researchers thus suggested generalized LL parsing techniques [5, 8, 11], which are more applicable to the larger grammar class than the class of LL grammars. On the other hand, the restricted LR(k) grammars, k-transformable grammars [3, 4], and PLR(k) grammars [10] were suggested by Hammer and Soisalon-Soininen, respectively, as the grammars able to be transformed into LL(k) covering grammars. Specially, k-transformable grammars have been conjectured as the uppermost class of such LR(k) grammars [3], and PLR(k) grammars have been known as a well-characterized subclass of k-transformable grammars [10].

In this paper we are interested in the covering transformation of LR grammars into LL grammars. The main idea in the transformation starts from the prediction of reduction goals during LR parsing. In LR parsing, a nonterminal, to which some prefix of the remaining input is reduced, is known at the reduction time, but we can often find a reduction goal before that time. A nonterminal B and a suffix γ of α are *predicted*, when the stack string is α, if it is certain that γ and some prefix of the remaining input will be reduced to B. (That is, the left side of Figure 1.1 is expected to be the right side.) In Hammer's method, γ is restricted to be ϵ, and in Soisalon-Soininen's method, γ has to be the left corner symbol of the right side of a production. Soisalon-Soininen's γ gives more information about B than Hammer's γ and, as a result, some predictable goals by Soisalon-Soininen's method cannot be

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predicted when Hammer's method is applied. We found that the lack of predictability causes the class of k-transformable grammars not to completely include the class of PLR(k) grammars.

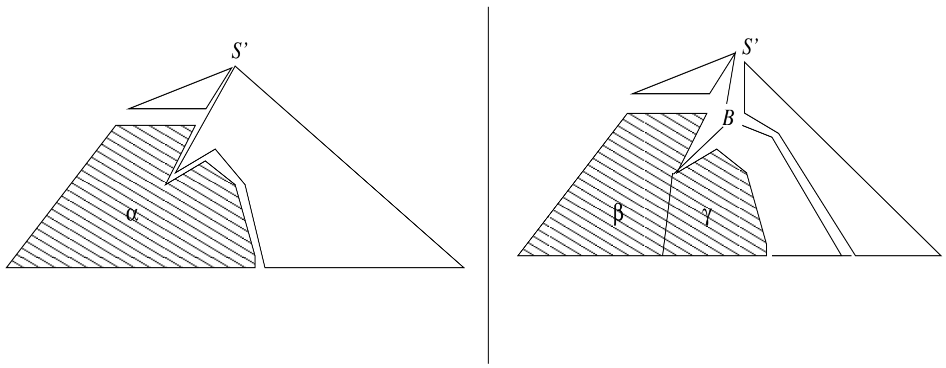


FIG. 1.1. *Prediction in LR parsing* ($\alpha = \beta\gamma$).

This paper generalizes the previous prediction technique into two viewpoints. The first key idea is to predict a reduction goal after enough information for the goal is known. The delay in predicting time obviously enlarges the range of predictable goals. In this paper, we allow γ in the right side of Figure 1.1 to be an arbitrary string. On the other hand, in both Hammer and Soisalon-Soininen's methods, the prediction of goal B at a parsing time implies that whenever a generatable string from B appears as the k-length prefix of the remaining input, a reduction to B will certainly occur. The second key idea is to allow the prediction to be performed not only over the total string but also over some partial string generatable from a predicted goal. That is, B can be a predicted goal as long as any predictive string exists, although some generatable strings from B do not guarantee a reduction to B.

This paper suggests a powerful covering transformation into LL(k) form based on the generalized prediction. The applicable grammar class is defined as *extended PLR(k) grammars*. They are larger than k-transformable grammars and PLR(k) grammars. Hammer informally described an extension [3, pp. 289–293] of k-transformable grammars. After his argument, there has been no more research on the extension. The main reason is considered to be intricateness of the construction of multiple stack machine [3], which is required to obtain an LL(k) transformation. (Actually, Hammer was worried that the extension needs numerous complications in his original machine model [3, p. 292, lines 1–3].) On the other hand, recently we suggested a grammatical characterization of k-transformable grammars in [7]. This paper develops an extension of k-transformable grammars using grammatical derivation. We believe that our extension completely includes the informal extension of Hammer.

Another contribution of this paper is the deterministic selection of the new transformation compared with the nondeterministic one of Hammer's method, where "deterministic" means that an LL covering grammar can be obtained in a single process, whereas in Hammer's method a transformer has to choose a set of predictable goals using a heuristic. As a result, whether a grammar is transformable or not can be decided in a single process. The grammatical characterization of the transformable grammars is also given.

Section 2 contains basic notation and definitions that are used in the subsequent sections. The counterexample showing that a PLR(k) grammar is not k-transformable is presented in section 3. Section 4 defines two relations, which are used to describe the generalized prediction. The idea for the deterministic transforming is developed in section 5. Section 6 suggests a new LL(k) covering transformation and relates the transformable grammars with k-transformable grammars and PLR(k) grammars. Finally, section 7 summarizes this paper.

2. Basic notation and definitions. We use notation and definitions based on [9], and the reader is assumed to be familiar with them.

Throughout this paper, the symbol G denotes an arbitrary context-free grammar $G = (N, \Sigma, P, S)$ (let $V = N \cup \Sigma$). The symbol k represents an arbitrary positive integer. Without being explicitly stated, G is assumed to be LR(k). Let $S' \notin N$ and a terminal symbol $\$ \notin \Sigma$. Then $G' = (N \cup \{S'\}, \Sigma \cup \{\$\}, P \cup \{S' \to S\$^k\}, S')$ is the augmented grammar of G. In this paper, G is assumed to be augmented and *reduced* [9].

Lowercase Greek letters, such as α, β, and γ, denote strings in V^*; lowercase Roman letters near the beginning of the alphabet, such as a, b, and c, are in Σ, and those near the end, such as w, x, y, and z, are in Σ^*; uppercase Roman letters near the end of the alphabet, such as W, X, and Y, are in V. The empty string is denoted by ϵ. The reverse of a string α is represented by α^R.

Let w be a terminal string. Then $k{:}w$ is equal to w if $|w| \leq k$, or the first k symbols of w otherwise. For any G and any $\alpha \in (V \cup \{\$\})^*$, $L^G(\alpha) = \{x | \alpha \Rightarrow^* x$ in G, $x \in \Sigma^*\}$, $FIRST_k^G(\alpha) = \{k{:}x | \alpha \Rightarrow^* x$ in G, $x \in \Sigma^*\}$, and $FOLLOW_k^G(\alpha) = \{k{:}x | S' \Rightarrow^* \beta\alpha x$ in G, $x \in \Sigma^*\}$. If G is obvious, then it is omitted.

A string $\alpha \in V^*$ is said to be a *viable prefix of* G if there exists a derivation $S' \Rightarrow_{rm}^* \beta B z \Rightarrow_{rm} \beta\gamma\delta z \Rightarrow_{rm}^* \beta\gamma y z$ in G, where $\beta\gamma = \alpha$ and rm stands for the rightmost derivation. For a viable prefix α of G, *k-right context* is defined as: $RC_k^G(\alpha) = \{k{:}yz|$ there exists $S' \Rightarrow_{rm}^* \beta B z \Rightarrow_{rm} \beta\gamma\delta z \Rightarrow_{rm}^* \beta\gamma y z$ in G where $\beta\gamma = \alpha\}$.

A restricted relation $\Rightarrow_{A,R}$ of $\Rightarrow_{rm}$, where $A \in N$ and $R \subseteq FOLLOW_k(A)$, is defined. Let $p = B \to X\delta$. Suppose that there exists $Ar \Rightarrow_{rm}^* \gamma B z r \Rightarrow_{rm}^p \gamma X \delta z r$, where $r \in R$ and $z \in \Sigma^*$. Then

$$\gamma B z r \Rightarrow_{A,R}^p \gamma X \delta z r \begin{cases} \text{holds when } FIRST_k(\delta z r) \cap R = \emptyset & \text{if } \gamma = \epsilon \text{ and } X = A, \\ \text{always holds} & \text{otherwise.} \end{cases}$$

Every derivation using $\Rightarrow_{A,R}$ starts from A and must not derive a string containing A at the leftmost position such that the string of the A immediately following belongs to R. That is, the derivation $Ar \Rightarrow_{A,R}^+ Axr$, where $r \in R$ and $k{:}xr \in R$ is not allowed. The difference between $\Rightarrow_{A,R}$ and $\Rightarrow_{rm}$ is clear when A is left recursive.

The k-right context function RC_k^G is localized over derivation $\Rightarrow_{A,R}$ as follows. Let $\alpha \in V^*$. Then $RC_k^{A,R}(\alpha) = \{k{:}yzr|$ there exists $Ar \Rightarrow_{A,R}^* \beta B z r \Rightarrow_{A,R} \beta\gamma\delta z r \Rightarrow_{A,R}^* \beta\gamma y z r$ in G where $r \in R$ and $\beta\gamma = \alpha\}$.

The equivalence of grammars can be considered in terms of languages or syntactic structures. That is, for G_1 and G_2, the former means $L(G_1) = L(G_2)$, and the latter means that the grammars satisfy a covering property. We present the definition of left-to-right cover.

DEFINITION 2.1. *Let* $G_1 = (N_1, \Sigma, P_1, S_1)$ *and* $G_2 = (N_2, \Sigma, P_2, S_2)$ *be grammars such that* $L(G_1) = L(G_2)$, *and let* h *be a homomorphism from* P_2 *to* P_1.

Then we say that G_2 left-to-right covers G_1 with respect to h if the following conditions are satisfied:

1. *If there exists $S_2 \Rightarrow_{lm}^{\pi} w$ in G_2, then there exists $S_1 \Rightarrow_{rm}^{h(\pi)} w$ in G_1, and*
2. *if there exists $S_1 \Rightarrow_{rm}^{\pi'} w$ in G_1, then there exists $S_2 \Rightarrow_{lm}^{\pi} w$ in G_2, where $h(\pi) = \pi'$.*

Let G_2 be LL(k) and G_1 be LR(k). Then G_2 *LL-to-LR(k) covers* G_1 if G_2 left-to-right covers G_1 with respect to h for some h. At this time, G_2 is said to be an *LL-to-LR*(k) *covering* (or simply, *LL*(k) *covering*) grammar of G_1.

We next present the definition of PLR(k) grammars.

Definition 2.2. *(see* [10]*). A grammar $G = (N, \Sigma, P, S)$ is said to be PLR(k) if in G', for each production $A \to X\delta$, where $X\delta \neq \epsilon$, the conditions*

$$S' \Rightarrow_{rm}^{*} \beta A z_1 \Rightarrow_{rm} \beta X \delta z_1 \Rightarrow_{rm}^{*} \beta X y_1 z_1,$$
$$S' \Rightarrow_{rm}^{*} \beta' B z_2 \Rightarrow_{rm} \beta' \beta'' X \zeta z_2 \Rightarrow_{rm}^{*} \beta' \beta'' X y_2 z_2, \text{ and}$$
$$\beta'\beta'' = \beta \text{ and } k{:}y_1 z_1 = k{:}y_2 z_2$$

always imply that $\beta A = \beta' B$.

Instead of the original definition [3] of k-transformable grammars, we use the following theorem to characterize k-transformable grammars because it does not require any understanding of the intricate multiple stack machine.

Theorem 2.3. *(see* [7]*). G is k-transformable iff there exists a constant n, depending on G, such that if $\alpha(\neq \epsilon)$ is a viable prefix of G and v is a string in $RC_k^G(\alpha)$, then there exist B, W, and $\gamma(\neq \epsilon)$, where $\alpha = \beta\gamma$, $|\gamma| \leq n$, and $v \in RC_k^{B,W}(\gamma)$ such that whenever there exists $S' \Rightarrow_{rm}^{*} \beta\gamma z \Rightarrow_{rm}^{*} \beta y z$ in G, where $k{:}yz \in RC_k^{B,W}(\epsilon)$, there exist $S' \Rightarrow_{rm}^{*} \beta B z''$ and $Bw \Rightarrow_{B,W}^{*} \gamma z' w \Rightarrow_{B,W}^{*} y z' w$ in G, where $w = k{:}z''(w \in W)$ and $z'z'' = z$. (We call the n the characteristic of k-transformableness for G.)*

3. A counterexample. The following example shows a PLR(1) grammar that is not k-transformable for all $k \geq 0$.

Example 3.1. Let $G1 = (\{S, C, D, X, Y\}, \{c, d, b, a\}, \{S \to C, C \to DXc, C \to DYd, D \to bD, D \to a, X \to DC, X \to DD, Y \to Dd\}, S)$. Then we can decide that $G1$ is LR(1) and PLR(1), but is not k-transformable for all $k (k \geq 1)$. Suppose that $G1$ is k-transformable for some fixed k and n is the characteristic of k-transformableness for $G1$. Let us set α and v to be $DD(DD)^j$ for some $j > n$ and b, respectively. Then we cannot find the B, W, and γ in Theorem 2.3. Hence, $G1$ cannot be k-transformable.

Thus, PLR(k) grammars are not a subclass of k-transformable grammars. Conversely, k-transformable grammars are not a subclass of PLR(k) grammars. For example, a grammar with the production set $\{S \to aAa, A \to Bb, B \to Aa, A \to b\}$ is LR(1) and 1-transformable. The grammar, however, is not PLR(1) because there exist two contradictory derivations $S' \Rightarrow_{rm} S\$ \Rightarrow_{rm} aAa\$$ and $S' \Rightarrow_{rm} S\$ \Rightarrow_{rm} aAa\$ \Rightarrow_{rm} aBba\$ \Rightarrow_{rm} aAaba\$$. Thus, there is no inclusion relationship between k-transformable grammars and PLR(k) grammars. A generalized grammar class, which contains both grammars, is hence motivated, and the subsequent sections treat this subject.

4. Two relations. This section defines two relations in order to formally describe the generalized prediction.

4.1. *d* relation. A rightmost derivation in a grammar can be analyzed through the dependency relation of Knuth [6]. The relation, called d relation, is here refined by attributing some terminal strings in order to represent context information in sentential forms.

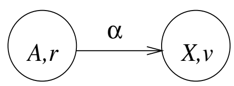
Let $A \in N$, $r \in \Sigma^k$, $\alpha \in V^*$, $X \in V \cup \{\epsilon\}$, and $v \in \Sigma^k \cup \Sigma^{k-1}$. Then

$$(4.1) \qquad (A, r)\ d^{\alpha}\ (X, v) \text{ iff } A \to \alpha X \beta \in P, r \in FOLLOW_k(A),$$

$$v \in FIRST_k(\beta r) \text{ when } X \in N \cup \{\epsilon\}, \text{ and } v \in FIRST_{k-1}(\beta r) \text{ when } X \in \Sigma.$$

Let $(A, r)\ d^{0^{\epsilon}}\ (A, r)$ be $(A, r)\ d^{\epsilon}\ (A, r)$, and $(A, r)\ d^{n^{\alpha}}\ (X, v)$, $n>0$, be the composition of $(A, r)\ d^{n-1^{\beta}}\ (B, w)$ and $(B, w)\ d^{\gamma}\ (X, v)$, where $\beta\gamma = \alpha$. The reflexive transitive closure of d relation, denoted by d^*, is defined by $\bigcup_{n \geq 0} d^n$.

The directed graph associated with the d relation is called the *d-graph*. It is constructed by representing each instance (4.1) as a pair of vertices connected by a directed edge:



A path is represented by $h = (A_0, r_0)\ d^{\alpha_1}\ (A_1, r_1) \cdots (A_{n-1}, r_{n-1})\ d^{\alpha_n}\ (A_n, r_n)$. If $A_n \in \Sigma$ or $A_n = \epsilon$, then h is a *terminal path*. If the intercomponents (A_1, r_1), …, (A_{n-1}, r_{n-1}) are not important, then h is represented by $(A_0, r_0)\ d^{*^{\alpha_1 \cdots \alpha_n}}\ (A_n, r_n)$.

Given a path in the d-graph, we can infer a corresponding rightmost derivation, and vice versa.

PROPERTY 4.1. *Let $A_i \in N$ and $r_i \in \Sigma^k$, $i = 0, 1, \ldots, n$. Then there exists a path $h = (A_0, r_0)\ d^{\alpha_1}\ (A_1, r_1)\ d^{\alpha_2}\ (A_2, r_2) \cdots (A_{n-1}, r_{n-1})\ d^{\alpha_n}\ (A_n, r_n)$ in the d-graph iff there exists a derivation in G such that $A_0 r_0 \Rightarrow_{rm} \alpha_1 A_1 \beta_1 r_0 \Rightarrow^*_{rm} \alpha_1 A_1 z_1 r_0 \Rightarrow_{rm} \alpha_1 \alpha_2 A_2 \beta_2 z_1 r_0 \Rightarrow^*_{rm} \alpha_1 \alpha_2 \cdots \alpha_{n-1} A_{n-1} z_{n-1} \cdots z_1 r_0 \Rightarrow_{rm} \alpha_1 \alpha_2 \cdots \alpha_{n-1} \alpha_n A_n \beta_n z_{n-1} \cdots z_1 r_0 \Rightarrow^*_{rm} \alpha_1 \alpha_2 \cdots \alpha_{n-1}\ \alpha_n A_n z_n z_{n-1} \cdots z_1 r_0$ for some $z_i \in \Sigma^*$ where $k{:}z_i \cdots z_1 r_0 = r_i$ $(i = 1, \ldots, n)$.*

Proof. The *only if* part can be proved by simple induction on n, and the *if* part can be proved by directly applying the definition of the d relation. □

A sequence of d^{ϵ}-related vertices in a path is defined as a *segment*. Let $h = (A_0, r_0) d^{\alpha_1}\ (A_1, r_1) \cdots (A_{n-1}, r_{n-1}) d^{\alpha_n} (A_n, r_n)$ be a path in the d-graph. If $\beta = \alpha_1 \cdots \alpha_i$ and $\alpha_{i+1} \cdots \alpha_j = \epsilon$, then the $|\beta|$-segment of h is $(A_i, r_i) d^{\alpha_{i+1}} (A_{i+1}, r_{i+1}) \cdots (A_{j-1}, r_{j-1})\ d^{\alpha_j}\ (A_j, r_j)$, where

$$(4.2) \qquad \alpha_i \neq \epsilon \text{ when } i \neq 0 \text{ and } \alpha_{j+1} \neq \epsilon \text{ when } j \neq n.$$

Condition (4.2) says that the immediate predecessor and the immediate successor of a segment, if they exist, are not d^{ϵ}-related.

PROPERTY 4.2. *Let $A \in N$ and $r \in \Sigma^k$. There exists $(A_l, r_l) = (A, r)$ for some l $(i \leq l \leq j)$ in the $|\beta|$-segment $(A_i, r_i)\ d^{\epsilon}\ (A_{i+1}, r_{i+1}) \cdots (A_{j-1}, r_{j-1}) d^{\epsilon}\ (A_j, r_j)$ of the path $(A_0, r_0) d^{*^{\alpha}} (A_n, r_n)$ iff there exists $A_0 r_0 \Rightarrow^*_{rm} \beta A z r_0$ in G, where $k{:}z r_0 = r$.*

We next define a specialized path in the d-graph that is related to a derivation over $\Rightarrow_{A,R}$.

DEFINITION 4.1. *Let $A \in N, r \in \Sigma^k, \alpha \in V^*$, and $u \in \Sigma^k$. Suppose that $h = (A_0, r_0)\ d^{\alpha_1}\ (A_1, r_1) \cdots (A_{n-1}, r_{n-1})\ d^{\alpha_n}\ (A_n, r_n)$ is a terminal path in the d-graph, where $A_0 = A, r_0 = r$, $\alpha_1 \cdots \alpha_n = \alpha$, and $A_n r_n = u$. Assume that there is no $i (1 \leq i \leq m)$ in the 0-segment $(A_0, r_0) d^{\epsilon}\ (A_1, r_1) \cdots d^{\epsilon}\ (A_m, r_m)$ of h such that $A_i = A$ and $r_i = r$. Then h is said to be an $\langle A, r, \alpha, u\rangle$-path.*

The following property can be obtained similarly to Property 4.1.

PROPERTY 4.3. *Let $r \in R$. If there exists an $\langle A, r, \alpha, u\rangle$-path in the d-graph, then there exists a derivation $Ar \Rightarrow^*_{A,R} \beta Bzr \Rightarrow_{A,R} \beta\gamma\delta zr \Rightarrow^*_{A,R} \beta\gamma yzr$ in G, where $\beta\gamma = \alpha$ and $k{:}yzr = u$, and vice versa.*

On the other hand, $RC_k^{A,R}$ can be obtained by examining some related paths in the d-graph as follows.

PROPERTY 4.4. $RC_k^{A,R}(\alpha) = \{u \mid$ *there exists an* $\langle A, r, \alpha, u\rangle$*-path in the d-graph where* $r \in R\}$.

Proof. ($\supseteq$) Assume that there exists an $\langle A, r, \alpha, u\rangle$-path in the d-graph, $h = (A, r)$ $d^{*^{\alpha}}$ (a, v), $r \in R$, where $av = u$. Then by Property 4.3, there exists a derivation $Ar \Rightarrow^*_{A,R} \beta Bzr \Rightarrow_{A,R} \beta\gamma\delta zr \Rightarrow^*_{A,R} \beta\gamma yzr$ in G, where $\beta\gamma = \alpha$ and $k{:}yzr = u$. Hence, $u \in RC_k^{A,R}(\alpha)$.

($\subseteq$) Let $u \in RC_k^{A,R}(\alpha)$. Then there exists a derivation $Ar \Rightarrow^*_{A,R} \beta Bzr \Rightarrow_{A,R} \beta\gamma\delta zr \Rightarrow^*_{A,R} \beta\gamma yzr$ in G, where $\beta\gamma = \alpha$ and $k{:}yzr = u$ for some $r \in R$. By Property 4.3, there exists an $\langle A, r, \alpha, u\rangle$-path in the d-graph. □

4.2. Π relation. We define a predictive relation, written by $\mathbf{\Pi}$, which represents predictable symbols based on the generalized idea.

DEFINITION 4.2. *Let $A, B \in N, R \subseteq FOLLOW_k(A), W \subseteq FOLLOW_k(B), \beta, \gamma \in V^*$, where $(A, R, \beta\gamma) \neq (B, W, \gamma)$ and $u \in RC_k^{B,W}(\gamma)$. Then $(A, R, \beta\gamma)\mathbf{\Pi}_u(B, W, \gamma)$ iff, for each $\langle A, r, \beta\gamma, u\rangle$-path, $r \in R$, we let $(A_0, r_0)d^\epsilon(A_1, r_1)\cdots(A_{n-1}, r_{n-1})d^\epsilon(A_n, r_n)$ be the $|\beta|$-segment of the path, then*

(i) *there exists m $(0 \leq m \leq n)$ such that*

$$A_m = B \tag{4.3}$$

and

$$\bigcup_h \{r_i | A_i = B, 0 \leq i \leq m\} \cap \bigcup_h \{r_i | A_i = B, m+1 \leq i \leq n\} = \emptyset, \tag{4.4}$$

where h in (4.4) *ranges over all $\langle A, r, \beta\gamma, u\rangle$-paths, and*

(ii) *if m_s is the smallest m ($m > 0$ when $\beta = \epsilon$) satisfying both* (4.3) *and* (4.4), *then*

$$W = \bigcup_h \{r_{m_s}\}, \tag{4.5}$$

where h in (4.5) *ranges over all $\langle A, r, \beta\gamma, u\rangle$-paths.*

$(A, R, \beta\gamma)\, \mathbf{\Pi}_u\, (B, W, \gamma)$ means that when the suffix $\beta\gamma$ of the current stack string is already predicted to be reduced to (A, R),[1] it is now predicted that whenever the k-length prefix of the remaining input is u, it is certain that the suffix γ of $\beta\gamma$ will be reduced to (B, W). Figure 4.1 depicts this prediction; the left tree is expected to be the right one. This relation expresses the generalized prediction; compared to Hammer and Soisalon-Soininen's methods, γ is not restricted, and the prediction is performed over a predictive string u rather than over predictive language [3].

The smallest condition in Definition 4.2(ii) is meaningful only when B is left recursive. If B is not left recursive, then the m in Definition 4.2(i) is unique, and so the smallest condition becomes meaningless.

[1] (A, R) represents a refinement of A with the context information R such that (A, R)'s following string belongs to R and the generated language from (A, R) is $L(A, R) = \{x | Ar \Rightarrow^*_{A,R} xr, r \in R\}$.

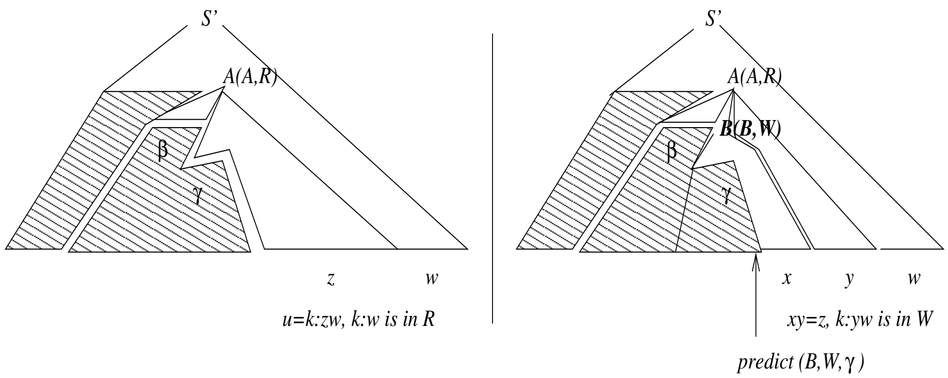


FIG. 4.1. *Predictive property.*

The following two lemmas delineate the meaning of $(A, R, \beta\gamma)\mathbf{II}_u(B, W, \gamma)$ in terms of grammatical derivation forms.

LEMMA 4.3. *Let* $(A, R, \beta\gamma)\mathbf{II}_u(B, W, \gamma)$. *Then whenever there exists* $Ar \Rightarrow^*_{A,R} \beta\gamma zr$, *where* $r \in R$ *and* $k{:}zr{=}u$, *there exist* $Ar \Rightarrow^*_{A,R} \beta Byr$ *and* $Bw \Rightarrow^*_{B,W} \gamma xw$, *where* $w = k{:}yr(w \in W)$ *and* $xy = z$.

Proof. Consider the derivation

$$B_0r \Rightarrow_{A,R} \beta_1B_1\delta_1r \Rightarrow^*_{A,R} \beta_1B_1z_1r \Rightarrow_{A,R} \beta_1\beta_2B_2\delta_2z_1r$$

(4.6)

$$\Rightarrow^*_{A,R} \cdots \Rightarrow^*_{A,R} \beta_1\beta_2\cdots\beta_nB_nz_n\cdots z_1r,$$

where $B_0 = A$, $\beta_1\cdots\beta_n = \beta\gamma$, and $k{:}zr = u$ with $B_nz_n\cdots z_1 = z$. Then there exists a corresponding path $h = (B_0, r_0)\, d^{\beta_1}\, (B_1, r_1)\, d^{\beta_2}\, (B_2, r_2)\ \cdots (B_{n-1}, r_{n-1})\, d^{\beta_n}\, (B_n, r_n)$, where $r_0 = r$, $r_i = k{:}z_i\cdots z_1r$ $(i = 1, \ldots, n-1)$, and $B_nr_n = u$. Here, h is an $\langle A, r, \beta\gamma, u\rangle$-path. Let $(A_0, r_0)d^\epsilon(A_1, r_1)\cdots d^\epsilon\ (A_n, r_n)$ be the $|\beta|$-segment of path h. By Definition 4.2 and (4.3), there exists A_m for some $m(0 \le m \le n)$ such that $A_m = B$, and by Definition 4.2 and (4.4), there is no $r_j \in W, m < j \le n$. Thus, the derivation (4.6) can be divided into $Ar \Rightarrow^*_{A,R} \beta Byr$, where $\beta = \beta_1\cdots\beta_m$ and $y = z_m\cdots z_1$ and $Bw \Rightarrow^*_{B,W} \gamma xw$, where $\gamma = \beta_{m+1}\cdots\beta_n$, $x = B_nz_n\cdots z_{m+1}$, and $w = k{:}yr = r_m$ $(w \in W)$. ◻

Similar to the proof of Lemma 4.3, we can prove the following lemma; the details are omitted.

LEMMA 4.4. *Let* $A, B \in N, R \subseteq FOLLOW_k(A)$, $W \subseteq FOLLOW_k(B)$, $\beta, \gamma \in V^*$, *and* $u \in RC_k^{B,W}(\gamma)$. *If, whenever there exists* $Ar \Rightarrow^*_{A,R} \beta\gamma zr$ *where* $r \in R$ *and* $k{:}zr{=}u$, *there exist* $Ar \Rightarrow^*_{A,R} \beta Byr$ *and* $Bw \Rightarrow^*_{B,W} \gamma xw$ *where* $w = k{:}yr(w \in W)$ *and* $xy = z$, *then* $(A, R, \beta\gamma)\mathbf{II}_u(B, W', \gamma)$ *for some* W'.

Let $(A, R, \beta\gamma)\mathbf{II}_u(B, W, \gamma)$ and $(A, R, \beta\gamma)\mathbf{II}_u(B, W', \gamma)$. Then W is equal to W' because of the smallest property in Definition 4.2 (ii).

The following example shows the computing process of $\mathbf{II}$ relations.

Example 4.1. Let $G2 = (\{S, A, C, B, X, Y\}, \{a, b, c\}, P2, S)$, where $P2 = \{S{\to}A$, $S{\to}C$, $A{\to}BX$, $A{\to}BY$, $C{\to}Ba$, $B{\to}b$, $X \to BA$, $X{\to}c$, $Y{\to}BC\}$. Note that $G2$ is LR(1). See $\langle A, \$, BBB, b\rangle$-paths and $\langle A, \$, BBB, c\rangle$-paths in Figure 4.2.

(i) We can find two $\langle A, \$, BBB, b\rangle$-paths, $p1 = (A, \$)\ d^B(X, \$)\ d^B\ (A, \$)\ d^B\ (X, \$)d^\epsilon(B, b)d^\epsilon(b, \epsilon)$ and $p2 = (A, \$)\ d^B(X, \$)\ d^B\ (A, \$)d^B\ (Y, \$)d^\epsilon(B, b)d^\epsilon(b, \epsilon)$. Then $p1$ and $p2$ have the same 2-segment $(A, \$)$. Hence, we have $(A, \{\$\}, BBB)\ \mathbf{II}_b\ (A, \{\$\}, B)$. On the other hand, $p1$ and $p2$ have the 3-segments $(X, \$)\ d^\epsilon\ (B, b)\ d^\epsilon\ (b, \epsilon)$ and $(Y, \$)\ d^\epsilon\ (B, b)\ d^\epsilon(b, \epsilon)$, respectively. Hence, we have $(A, \{\$\}, BBB)\ \mathbf{II}_b\ (B, \{b\}, \epsilon)$.

(ii) We next find one $\langle A, \$, BBB, c\rangle$-path, $p3 = (A, \$)\ d^B(X, \$)\ d^B\ (A, \$)\ d^B\ (X, \$)\ d^\epsilon\ (c, \epsilon)$. By examining this path, we have $(A, \{\$\}, BBB)\ \mathbf{II}_c\ (A, \{\$\}, B)$ and $(A, \{\$\}, BBB)\ \mathbf{II}_c\ (X, \{\$\}, \epsilon)$.

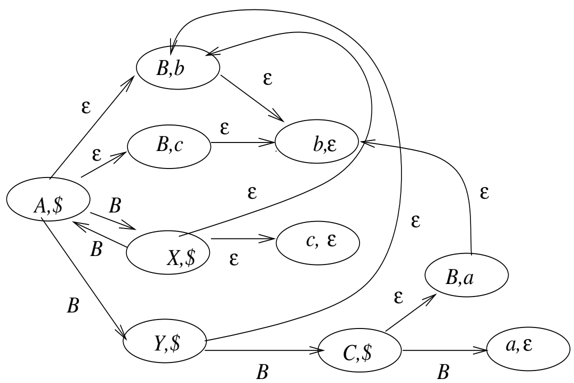


FIG. 4.2. *A part of the d-graph.*

In the computation of the $\mathbf{II}$ relation, visiting vertices in the d-graph a bounded number of times is enough even when cycles are present in the d-graph.

5. Deterministic prediction. This section develops a deterministic procedure for choosing only one relation when there exist more than one $\mathbf{II}_u$ relation with a fixed left side. In Hammer's method, a heuristic is used for the selection. In this paper, the choice problem is resolved by choosing the nearer one in the derivation tree between the corresponding positions of (B, W, γ) and (C, X, ζ) to that of (A, R, α). Let $\alpha = \beta\gamma$ and $\gamma = \delta\zeta$. The roots $(A, R), (B, W)$, and (C, X) of the subtrees constructed from $(A, R, \alpha), (B, W, \gamma)$, and (C, X, ζ) are shown in Figure 5.1. In this situation, (B, W, γ) is selected since (A, R) is nearer to (B, W) than it is to (C, X).

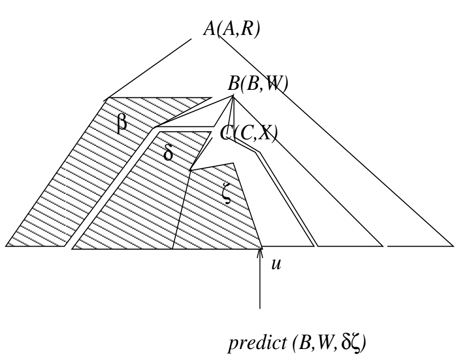


FIG. 5.1. *Derivation tree.*

In the remaining part of the paper, if $(A, R, \beta\gamma)\mathbf{\Pi}_u(B, W, \gamma)$, then we assume $u \in \{k{:}x | Bw \Rightarrow^*_{B,W} \gamma xw, w \in W\}$ for some technical simplification.

The following lemma is used to show the orderable property of $\mathbf{\Pi}$ in the nearest sequence.

LEMMA 5.1.

(i) *Let* $(A, R, \beta\delta\zeta)\mathbf{\Pi}_u(B, W, \delta\zeta)$ *and* $(A, R, \beta\delta\zeta)\mathbf{\Pi}_u(C, X, \zeta)$. *Then for each* $w \in W$, *there exists* $x \in X$ *such that* $(B, w){d^*}^{\delta}(C, x)$.

(ii) *Let* $(A, R, \beta\delta\zeta)$ $\mathbf{\Pi}_u$ $(B, W, \delta\zeta)$ *and* $(B, W, \delta\zeta)$ $\mathbf{\Pi}_u$ (C, X, ζ). *Then* $(A, R, \beta\delta\zeta)$ $\mathbf{\Pi}_u$ (C, X, ζ).

(iii) *Let* $(A, R, \beta\delta\zeta)\mathbf{\Pi}_u(B, W, \delta\zeta)$ *and* $(A, R, \beta\delta\zeta)\mathbf{\Pi}_u(C, X, \zeta)$, *where* $(B, W, \delta\zeta) \neq (C, X, \zeta)$. *Then when* $\delta \neq \epsilon$, $(B, W, \delta\zeta)\mathbf{\Pi}_u(C, X, \zeta)$; *when* $\delta = \epsilon$, $(B, W, \zeta)\mathbf{\Pi}_u(C, X, \zeta)$ *or* $(C, X, \zeta)\mathbf{\Pi}_u(B, W, \zeta)$.

(iv) *There are no* $A, R, \alpha, u, B, W, \gamma$ *such that* $(A, R, \alpha)\mathbf{\Pi}_u(B, W, \gamma)$ *and* (B, W, γ) $\mathbf{\Pi}_u$ (A, R, α), *where* $(A, R, \alpha) \neq (B, W, \gamma)$.

Proof.

(i) Let $w \in W$. Then there exists an $\langle A, r, \beta\delta\zeta, u\rangle$-path h, where $r \in R$ which contains (B, w) on the $|\beta|$-segment. The path h also contains (C, x) for some $x \in X$ on the $|\beta\delta|$-segment. Hence, $(B, w){d^*}^{\delta}(C, x)$ holds, and Lemma 5.1(i) is true.

(ii) Let $r \in R$. Take an arbitrary $\langle A, r, \beta\delta\zeta, u\rangle$-path h. Since $(A, R, \beta\delta\zeta)\mathbf{\Pi}_u(B, W, \delta\zeta)$ holds, h contains (B, w) for some $w \in W$ on the $|\beta|$-segment. Take an arbitrary $\langle B, w, \delta\zeta, u\rangle$-path $h2$ that is a suffix of h. Since $(B, W, \delta\zeta)\mathbf{\Pi}_u(C, X, \zeta)$ holds, $h2$ contains (C, x) for some $x \in X$ on the $|\delta|$-segment. Hence, $(A, R, \beta\delta\zeta)\mathbf{\Pi}_u(C, X', \zeta)$ holds. The next goal is to show $X' = X$. When $\delta \neq \epsilon$, it is obvious that $X' = X$. We next consider the case of $\delta = \epsilon$. Figure 5.2(a)–(c) shows the possible forms of an $\langle A, r, \beta\delta\zeta, u\rangle$-path containing (B, w) and (C, x) on the $|\beta|$-segment. By investigating the figure, we know that there are some paths in (a) and (c) that do not contain (B, w) or (C, x), and so (b) is the only correct form. Hence, we can conclude that every (C, x), $x \in X$ on the $|\beta|$-segment of h follows (B, w) for some $w \in W$ on the same segment. As a consequence, X' has to be equal to X.

(iii) We first think about the case of $\delta \neq \epsilon$. Let $w \in W$. Take an arbitrary $\langle B, w, \delta\zeta, u\rangle$-path h. Then according to Lemma 5.1(i), h has a $\langle C, x, \zeta, u\rangle$-path for some $x \in X$ as a suffix. Hence, $(B, W, \delta\zeta)\mathbf{\Pi}_u(C, X, \zeta)$ holds. Let us consider the other case of $\delta = \epsilon$. Then either one of $(B, W, \zeta)\mathbf{\Pi}_u(C, X', \zeta)$ or $(C, X, \zeta)\mathbf{\Pi}_u(B, W', \zeta)$ holds. Without loss of generality, we assume $(B, W, \zeta)\mathbf{\Pi}_u(C, X', \zeta)$. At this point, by Lemma 5.1(ii), $(A, R, \beta\delta\zeta)\mathbf{\Pi}_u(C, X', \zeta)$ holds. On the other hand, X' is uniquely determined, and thus $X' = X$. In all, $(B, W, \zeta)\mathbf{\Pi}_u(C, X, \zeta)$ holds.

(iv) Assume that $(A, R, \alpha)\mathbf{\Pi}_u(B, W, \gamma)$ and $(B, W, \gamma)\mathbf{\Pi}_u(A, R, \alpha)$, where $(A, R, \alpha) \neq (B, W, \gamma)$. Then $\alpha = \gamma$, and $(A, R) \neq (B, W)$ because of $(A, R, \alpha) \neq (B, W, \gamma)$. Take an arbitrary $\langle B, w, \gamma, u\rangle$-path h, $w \in W$. Then h is a suffix of an $\langle A, r, \alpha, u\rangle$-path for some $r \in R$ because of $(A, R, \alpha)\mathbf{\Pi}_u(B, W, \gamma)$. Next, take an arbitrary $\langle A, r, \alpha, u\rangle$-path $h2$; then, $h2$ is a suffix of a $\langle B, w, \gamma, u\rangle$-path for some $w \in W$ because of $(B, W, \gamma)\mathbf{\Pi}_u(A, R, \alpha)$. However, the 0-segment of an $\langle A, r, \alpha, u\rangle$-path cannot contain (A, r) in any other position but the first. As a result, both $(A, R, \alpha)\mathbf{\Pi}_u(B, W, \gamma)$ and $(B, W, \gamma)\mathbf{\Pi}_u(A, R, \alpha)$ cannot be true. □

Let $(A, R, \alpha)\mathbf{\Pi}_u(B, W, \gamma)$ and $(A, R, \alpha)\mathbf{\Pi}_u(C, X, \zeta)$. Then by Lemma 5.1(iii), we can assume $(B, W, \gamma)\mathbf{\Pi}_u(C, X, \zeta)$ without loss of generality, and then, by Lemma 5.1(iv), $(C, X, \zeta)\mathbf{\Pi}_u(B, W, \gamma)$ does not hold. By generalizing this property, we have the following lemma.

LEMMA 5.2. *Let* $(A, R, \alpha)\mathbf{\Pi}_u(B_0, W_0, \gamma_0)$, $(A, R, \alpha)\mathbf{\Pi}_u(B_1, W_1, \gamma_1)$, …, (A, R, α)

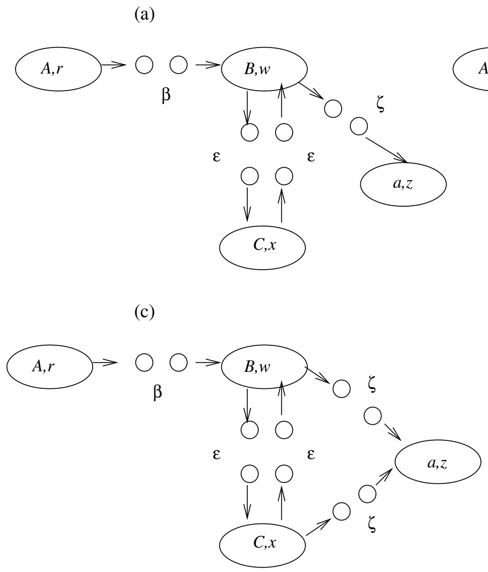


FIG. 5.2. *Paths in the d-graph, where* $az = u$.

$\mathbf{III}_u$ (B_n, W_n, γ_n) *be distinct relations. For the elements* $(B_0, W_0, \gamma_0), (B_1, W_1, \gamma_1), \ldots,$ (B_n, W_n, γ_n)*, there exists a unique sequence* $(B_{m_0}, W_{m_0}, \gamma_{m_0})$, $(B_{m_1}, W_{m_1}, \gamma_{m_1})$, $\ldots$, $(B_{m_n}, W_{m_n}, \gamma_{m_n})$ *such that* $(B_{m_{i-1}}, W_{m_{i-1}}, \gamma_{m_{i-1}})\ \mathbf{III}_u\ (B_{m_i}, W_{m_i}, \gamma_{m_i})$, $i = 1, \ldots, n$.

The unique ordering property in Lemma 5.2 enable us to linearize $\mathbf{III}$ relations. Consequently, we can define $\mathbf{\Phi}$ function as follows.

DEFINITION 5.3. *Let* A, R, α, u *be given. For all the distinct elements* (B_0, W_0, γ_0), (B_1, W_1, γ_1), $\ldots$, (B_n, W_n, γ_n) *such that* $(A, R, \alpha)\mathbf{III}_u(B_i, W_i, \gamma_i)$, $i = 0, \ldots, n$, *let* $(B_{m_0}, W_{m_0}, \gamma_{m_0})$, $(B_{m_1}, W_{m_1}, \gamma_{m_1})$, $\ldots$, $(B_{m_n}, W_{m_n}, \gamma_{m_n})$ *be the sequence such that* $(B_{m_{i-1}}, W_{m_{i-1}}, \gamma_{m_{i-1}})\mathbf{III}_u(B_{m_i}, W_{m_i}, \gamma_{m_i})$, $i = 1, \ldots, n$. *Then* $\mathbf{\Phi}(A, R, \alpha, u)$ *is defined as* $(B_{m_0}, W_{m_0}, \gamma_{m_0})$; *otherwise,* $\mathbf{\Phi}(A, R, \alpha, u)$ *is undefined.*

We have, in Example 4.1, $(A, \{\$\}, BBB)\ \mathbf{III}_b\ (A, \{\$\}, B)$ and $(A, \{\$\}, BBB)\ \mathbf{III}_b\ (B, \{\$\}, \epsilon)$. Then we have the unique sequence $(A, \{\$\}, B)$, $(B, \{\$\}, \epsilon)$ satisfying the condition in Lemma 5.2; note that $(A, \{\$\}, B)\mathbf{III}_b(B, \{\$\}, \epsilon)$ holds. Hence, $\mathbf{\Phi}(A, \{\$\}, BBB, b)$ is defined as $(A, \{\$\}, B)$. Similarly, $\mathbf{\Phi}(A, \{\$\}, BBB, c)$ is defined as $(A, \{\$\}, B)$.

6. Extended PLR(k) grammars. Using the $\mathbf{\Phi}$ function, we give a transformation into LL(k) form, and the transformable grammars are defined as extended PLR(k) grammars.

6.1. A transformation. Given G, the following algorithm is applied to obtain $\boldsymbol{T}(G)$.

ALGORITHM 1 (*construction of* $\boldsymbol{T}(G)$).

INPUT: G

OUTPUT: If this algorithm successfully terminates, $\boldsymbol{T}(G) = (\boldsymbol{N}, \Sigma, \boldsymbol{P}, \boldsymbol{S})$ *is constructed.*

METHOD:

1. *Initially,*

$\boldsymbol{S} = [S, \{\$^k\}, \epsilon, FIRST_k(S\$^k)]$; $\boldsymbol{N} = \{\boldsymbol{S}\}$; $\boldsymbol{P} = \emptyset$.

2. **repeat**
   (a) *For each* $[A, R, \alpha, U] \in \boldsymbol{N}$, *let* $Z = \{u \in U \mid \boldsymbol{\Phi}(A, R, \alpha, u)$ *is defined*$\}$, *then do the following*:
   Type 1. $\boldsymbol{P} = \boldsymbol{P} \cup \{[A, R, \alpha, U] \to a[A, R, \alpha a, V]\}$ *if* $V \neq \emptyset$, *where* $V = \{k{:}zwr \mid$ *there exists* $Ar \Rightarrow^*_{A,R} \beta Bwr \Rightarrow_{A,R} \beta\gamma a\delta wr \Rightarrow^*_{A,R} \beta\gamma azwr$ *in* G, *where* $r \in R$, $\beta\gamma = \alpha$, $k{:}azwr \in U$, *and* $k{:}azwr \notin Z\}$.
   Type 2. $\boldsymbol{P} = \boldsymbol{P} \cup \{[A, R, \alpha, U] \to [A, R, \beta B, V]\}$, *where* $\alpha = \beta\gamma$ *if* $V \neq \emptyset$, *where* $V = \{k{:}wr \mid$ *there exists* $Ar \Rightarrow^*_{A,R} \beta Bwr \Rightarrow_{A,R} \beta\gamma wr$ *in* G, *where* $r \in R$, $k{:}wr \in U$, *and* $k{:}wr \notin Z\}$.
   Type 3. $\boldsymbol{P} = \boldsymbol{P} \cup \{[A, R, A, U] \to \epsilon\}$, *where* $R \cap U \neq \emptyset$ *and* $\alpha = A$.
   Type 4. $\boldsymbol{P} = \boldsymbol{P} \cup \{[A, R, \alpha, U] \to [B, W, \gamma, V][A, R, \beta B, W]\}$, *where* $\alpha = \beta\gamma$ *if* $V \neq \emptyset$, *where* $V = \{u \mid \boldsymbol{\Phi}(A, R, \alpha, u) = (B, W, \gamma)$, $u \in U\}$.
   (b) *New nonterminals that appeared in* $\boldsymbol{P}$ *are added to* $\boldsymbol{N}$.

   until(*$\boldsymbol{P}$ is not changed*)

Algorithm 1 does not successfully terminate when an infinite number of nonterminals is generated. The formal characterization of the successfully transformable grammars is given in section 6.3. The remarkable observation is that our transformation depends on only G while Hammer's transformation depends on both G and a cycle-free multiple stack machine for G. As a result, given G, $\boldsymbol{T}(G)$ is constructed in a single process.

6.2. The LL(k) covering property. A homomorphism h is defined from $\boldsymbol{P}$ to $P \cup \{\epsilon\}$:

$$h(p_T) = \begin{cases} B \to \gamma & \text{if } p_T = [A, R, \beta\gamma, U] \to [A, R, \beta B, V], \\ \epsilon & \text{otherwise.} \end{cases}$$

The following two lemmas show a relationship between a rightmost derivation in G and a leftmost derivation in $\boldsymbol{T}(G)$.

LEMMA 6.1. *Let* $[A, R, \alpha, U] \in \boldsymbol{N}$. *Assume that there exists* π *such that* $Ar \Rightarrow^{\pi}_{A,R} \alpha xr$ *in* G, *where* $r \in R$ *and* $k{:}xr \in U$. *Then there exists* π_T *such that* $[A, R, \alpha, U] \Rightarrow^{\pi_T}_{lm} x$ *in* $\boldsymbol{T}(G)$, *where* $h(\pi_T) = \pi^R$.

Proof. We use induction on $|\pi|$. As the basis, assume that $|\pi| = 1$. Then $A \to \alpha x \in P$, and there exist an $|x|$-length string π_T of Type 1 rules such that $[A, R, \alpha, U] \Rightarrow^{\pi_T}_{lm} x[A, R, \alpha x, T]$ and a rule string $p^1_T p^2_T$ such that $[A, R, \alpha x, T] \Rightarrow^{p^1_T}_{lm} [A, R, A, M] \Rightarrow^{p^2_T}_{lm} \epsilon$ in $\boldsymbol{T}(G)$ for some M. Here, $h(\pi_T p^1_T p^2_T) = h(\pi_T)h(p^1_T)h(p^2_T) = h(p^1_T) = A \to \alpha x$ because of $h(\pi_T) = \epsilon$ and $h(p^2_T) = \epsilon$. Hence, this lemma holds for the basis step. As an inductive hypothesis, assume that this lemma holds when $|\pi| \leq l (l > 1)$. Now suppose that $|\pi| > l$.

Case 1. There exist y, B, W, and γ such that $\boldsymbol{\Phi}(A, R, \alpha y, u) = (B, W, \gamma)$, where $u = k{:}zr$ with $yz = x$.

In this case, we have a $|y|$-length string π'_T of Type 1 rules such that $[A, R, \alpha, U] \Rightarrow^{\pi'_T}_{lm} y[A, R, \alpha y, X]$, where $u \in X$ and a rule p_T such that $[A, R, \alpha y, X] \Rightarrow^{p_T}_{lm} [B, W, \gamma, V][A, R, \beta B, W]$, where $u \in V$ and $\alpha y = \beta\gamma$ in $\boldsymbol{T}(G)$. On the other hand, π such that $Ar \Rightarrow^{\pi}_{A,R} \alpha yzr$ can be divided into π'' and π''' such that $Ar \Rightarrow^{\pi''}_{A,R} \beta Bz_2 r$ and $Bw \Rightarrow^{\pi'''}_{B,W} \gamma z_1 w$, where $w = k{:}z_2 r (w \in W)$ and $z_1 z_2 = z$ by Lemma 4.3. Note that $k{:}z_1 w \in V$ and $k{:}z_2 r \in W$. By applying the inductive hypothesis to π'' and π''', we have $[A, R, \beta B, W] \Rightarrow^{\pi''_T}_{lm} z_2$ and $[B, W, \gamma, V] \Rightarrow^{\pi'''_T}_{lm} z_1$ in $\boldsymbol{T}(G)$, where $h(\pi''_T) = (\pi'')^R$ and $h(\pi'''_T) = (\pi''')^R$. Therefore, there exists a rule string $\pi_T (= \pi'_T p_T \pi'''_T \pi''_T)$ such that $[A, R, \alpha, U] \Rightarrow_{lm}^{\pi_T} yz(=x)$ in $\boldsymbol{T}(G)$. Here, $h(\pi_T) = h(\pi'_T p_T \pi'''_T \pi''_T) = h(\pi'''_T)h(\pi''_T) = (\pi''')^R(\pi'')^R = (\pi''\pi''')^R = \pi^R$ because of $h(\pi'_T) = \epsilon$ and $h(p_T) = \epsilon$.

Case 2. The other cases.

(Subcase 2-1) The π is composed of π' and p such that $Ar \Rightarrow_{A,R}^{\pi'} \beta Bzr \Rightarrow_{A,R}^{p} \beta\gamma yzr$, where $\beta\gamma = \alpha$ and $yz = x$. Then there exist a $|y|$-length string π''_T of Type 1 rules such that $[A, R, \alpha, U] \Rightarrow_{lm}^{\pi''_T} y[A, R, \alpha y, M]$, where $k{:}zr \in M$, and a Type 2 rule p_T such that $[A, R, \alpha y, M] \Rightarrow_{lm}^{p_T} [A, R, \beta B, W]$, where $k{:}zr \in W$ in $\boldsymbol{T}(G)$ according to the construction of $\boldsymbol{T}(G)$. On the other hand, by applying the inductive hypothesis to π', we have $[A, R, \beta B, W] \Rightarrow_{lm}^{\pi'_T} z$ in $\boldsymbol{T}(G)$, where $h(\pi'_T) = (\pi')^R$. In all, there exists $\pi_T(= \pi''_T p_T \pi'_T)$ such that $[A, R, \alpha, U] \Rightarrow_{lm}^{\pi_T} yz$ in $\boldsymbol{T}(G)$, where $yz = x$. Here, $h(\pi_T) = h(\pi''_T p_T \pi'_T) = h(\pi''_T)h(p_T)h(\pi'_T) = h(p_T)h(\pi'_T) = ph(\pi'_T) = p\pi'^R = (\pi' p)^R = \pi^R$ because of $h(\pi''_T) = \epsilon$.

(Subcase 2-2) The π is composed of π' and p such that $Ar \Rightarrow_{A,R}^{\pi'} \alpha y_1 Bzr \Rightarrow_{A,R}^{p} \alpha y_1 y_2 zr$, where $y_1 y_2 z = x$. Then there exist a $|y_1 y_2|$-length string π''_T of Type 1 rules such that $[A, R, \alpha, U] \Rightarrow_{lm}^{\pi''_T} y_1 y_2 [A, R, \alpha y_1 y_2, M]$, where $k{:}zr \in M$, and a Type 2 rule p_T such that $[A, R, \alpha y_1 y_2, M] \Rightarrow_{lm}^{p_T} [A, R, \alpha y_1 B, W]$, where $k{:}zr \in W$ in $\boldsymbol{T}(G)$. On the other hand, we have $[A, R, \alpha y_1 B, W] \Rightarrow_{lm}^{\pi'_T} z$ in $\boldsymbol{T}(G)$, where $h(\pi'_T) = (\pi')^R$ by applying the inductive hypothesis to π'. In all, there exists $\pi_T(= \pi''_T p_T \pi'_T)$ such that $[A, R, \alpha, U] \Rightarrow_{lm}^{\pi_T} y_1 y_2 z$ in $\boldsymbol{T}(G)$. Here, $h(\pi_T) = h(\pi''_T p_T \pi'_T) = h(\pi''_T)h(p_T)h(\pi'_T) = h(p_T)h(\pi'_T) = ph(\pi'_T) = p(\pi')^R = (\pi' p)^R = \pi^R$ because of $h(\pi''_T) = \epsilon$. ◻

LEMMA 6.2. *If there exists π_T such that $[A, R, \alpha, U] \Rightarrow_{lm}^{\pi_T} x$ in $\boldsymbol{T}(G)$, then there exists $Ar \Rightarrow_{A,R}^{h(\pi_T)^R} \alpha xr$ in G, where $r \in R$ and $k{:}xr \in U$.*

Proof. If $|\pi_T| = 1$, then $x = \epsilon, \alpha = A$, and $R \cap U \neq \emptyset$. Let $r \in R \cap U$; then there exists $Ar \Rightarrow_{A,R}^{\epsilon} Ar$ in G. Note that $k{:}xr \in U$ because of $k{:}xr = k{:}r = r$ and $r \in U$; $h(\pi_T)^R = \epsilon^R = \epsilon$. We have completed the proof of the basis case. Next, as an inductive hypothesis, assume that this lemma holds when $|\pi_T| \leq l(l > 1)$. Now let $|\pi_T| > l$ and $\pi_T = p_T \pi'_T$. Then p_T is a rule of Type 1, Type 2, or Type 4. According to the type, we divide the cases as follows.

Case 1. $p_T = [A, R, \alpha, U] \to a[A, R, \alpha a, Q]$.

Let $x = az$. Then there exists π'_T such that $[A, R, \alpha a, Q] \Rightarrow_{lm}^{\pi'_T} z$ in $\boldsymbol{T}(G)$. By applying the inductive hypothesis to π'_T, we have $Ar \Rightarrow_{A,R}^{h(\pi'_T)^R} \alpha azr$ in G, where $r \in R$ and $k{:}zr \in Q$. According to the construction of the set Q, $k{:}azr \in U$. Note that $h(\pi_T)^R = h(p_T \pi'_T)^R = h(\pi'_T)^R$ because of $h(p_T) = \epsilon$.

Case 2. $p_T = [A, R, \alpha, U] \to [A, R, \beta B, W]$.

Let $\alpha = \beta\gamma$ and $p = B \to \gamma$. Then there exists π'_T such that $[A, R, \beta B, W] \Rightarrow_{lm}^{\pi'_T} x$ in $\boldsymbol{T}(G)$, and p is a rule in P. By applying the inductive hypothesis to π'_T, we have $Ar \Rightarrow_{A,R}^{h(\pi'_T)^R} \beta Bxr$ in G, where $r \in R$ and $k{:}xr \in W$. On the other hand, the condition of $k{:}xr \in U$ is true according to $\boldsymbol{T}(G)$'s construction. Hence, we have the derivation $Ar \Rightarrow_{A,R}^{h(\pi'_T)^R} \beta Bxr \Rightarrow_{A,R}^{p} \beta\gamma xr$ in G, where $r \in R$ and $k{:}xr \in U$. Here, $h(\pi'_T)^R p = (ph(\pi'_T))^R = (h(p_T)h(\pi'_T))^R = h(p_T \pi'_T)^R = h(\pi_T)^R$.

Case 3. $p_T = [A, R, \alpha, U] \to [B, W, \gamma, Y][A, R, \beta B, W]$.

Let $[B, W, \gamma, Y] \Rightarrow_{lm}^{\pi''_T} x_1$ and $[A, R, \beta B, W] \Rightarrow_{lm}^{\pi'''_T} x_2$ in $\boldsymbol{T}(G)$, where $x_1 x_2 = x$ and $p_T \pi''_T \pi'''_T = \pi_T$. Then by applying the inductive hypothesis to π''_T and π'''_T, we have $Bw \Rightarrow_{B,W}^{h(\pi''_T)^R} \gamma x_1 w$, where $w \in W$ and $k{:}x_1 w \in Y$, and $Ar \Rightarrow_{A,R}^{h(\pi'''_T)^R} \beta Bx_2 r$, where $r \in R$ and $k{:}x_2 r \in W$ in G. Here, $k{:}x_1 w \in U$ because of $Y \subseteq U$. Without loss of generality, we set w to be $k{:}x_2 r$. Then we have $Ar \Rightarrow_{A,R}^{h(\pi'''_T)^R} \beta Bx_2 r \Rightarrow_{A,R}^{h(\pi''_T)^R} \beta\gamma x_1 x_2 r$ in G. Here, $k{:}x_1 x_2 r = k{:}x_1(k{:}x_2 r) = k{:}x_1 w$, and so $k{:}x_1 x_2 r \in U$. Note that

$h(\pi'''_T)^R h(\pi''_T)^R = (h(\pi''_T)h(\pi'''_T))^R = (h(p_T)h(\pi''_T)h(\pi'''_T))^R = h(p_T\pi''_T\pi'''_T)^R = h(\pi_T)^R$ because of $h(p_T) = \epsilon$.

In all, we have this lemma. □

From Lemmas 6.1 and 6.2, we get the following corollaries.

COROLLARY 6.3. *If there exists* π *such that* $S\$^k \Rightarrow^{\pi}_{rm} x\$^k$ *in* G*, then there exists* π_T *such that* $[S, \{\$^k\}, \epsilon, FIRST_k(S\$^k)] \Rightarrow^{\pi_T}_{lm} x$ *in* $\boldsymbol{T}(G)$*, where* $h(\pi_T) = \pi^R$*.*

Proof. If there exists π such that $S\$^k \Rightarrow^{\pi}_{rm} x\$^k$ in G, then there exists $S\$^k \Rightarrow^{\pi}_{S,\{\$^k\}} x\$^k$ in G. On the other hand, $[S, \{\$^k\}, \epsilon, FIRST_k(S\$^k)] \in \boldsymbol{N}$. Then according to Lemma 6.1, there exists π_T such that $[S, \{\$^k\}, \epsilon, FIRST_k(S\$^k)] \Rightarrow^{\pi_T}_{lm} x$ in $\boldsymbol{T}(G)$, where $h(\pi_T) = \pi^R$. □

COROLLARY 6.4. *If there exists* π_T *such that* $[S, \{\$^k\}, \epsilon, FIRST_k(S\$^k)] \Rightarrow^{\pi_T}_{lm} x$ *in* $\boldsymbol{T}(G)$*, then there exists* $S\$^k \Rightarrow^{h(\pi_T)^R}_{rm} x\$^k$ *in* G*.*

Proof. By Lemma 6.2, if there exists π_T such that $[S, \{\$^k\}, \epsilon, FIRST_k(S\$^k)] \Rightarrow^{\pi_T}_{lm} x$ in $\boldsymbol{T}(G)$, then there exists $S\$^k \Rightarrow^{h(\pi_T)^R}_{S,\{\$^k\}} x\$^k$ in G, and so there exists $S\$^k \Rightarrow^{h(\pi_T)^R}_{rm} x\$^k$ in G. □

According to Corollaries 6.3 and 6.4, we conclude the following theorem.

THEOREM 6.5. $\boldsymbol{T}(G)$ *left-to-right covers* G *with respect to* h*.*

On the other hand, the LL property of $\boldsymbol{T}(G)$ can be proved as follows.

THEOREM 6.6. $\boldsymbol{T}(G)$ *is LL(*k*).*

Proof. Suppose that $\boldsymbol{T}(G)$ is not LL(k). Then there exist two derivations in $\boldsymbol{T}(G)$ such that

$$[S, \{\$^k\}, \epsilon, FIRST_k(S\$^k)] \Rightarrow^*_{lm} x[A, R, \alpha, U]\tau \Rightarrow_{lm} x\omega\tau \Rightarrow^*_{lm} xy\tau \Rightarrow^*_{lm} xyz$$

and

$$[S, \{\$^k\}, \epsilon, FIRST_k(S\$^k)] \Rightarrow^*_{lm} x[A, R, \alpha, U]\tau' \Rightarrow_{lm} x\rho\tau' \Rightarrow^*_{lm} xy'\tau' \Rightarrow^*_{lm} xy'z',$$

where $k{:}yz = k{:}y'z'$ and $\omega \neq \rho$. Here, we can observe that $k{:}z\$^k \in R$ and $k{:}z'\$^k \in R$. Let r and r' denote $k{:}z\$^k$ and $k{:}z'\$^k$, respectively. At this time, we have $k{:}yr = k{:}y'r'$. Denote $[A, R, \alpha, U] \to \omega$ and $[A, R, \alpha, U] \to \rho$ by p^1_T and p^2_T, respectively. The possible situations for p^1_T and p^2_T are divided as follows.

Case 1. $p^1_T = [A, R, \alpha, U] \to a[A, R, \alpha a, V]$.

Then there exists $[A, R, \alpha a, V] \Rightarrow^*_{lm} y_1$ in $\boldsymbol{T}(G)$, where $ay_1 = y$. By applying Lemma 6.2 to this derivation, we have

$$Ar \Rightarrow^*_{A,R} \alpha a y_1 r \text{ in } G. \tag{6.1}$$

Derivation (6.1) is of the form

$$Ar \Rightarrow^*_{A,R} \beta B w r \Rightarrow_{A,R} \beta\gamma a\delta w r \Rightarrow^*_{A,R} \beta\gamma a y_1 r, \tag{6.2}$$

where $\beta\gamma = \alpha$ according to Type 1 rule's construction.

(Subcase 1-1) $p^2_T = [A, R, \alpha, U] \to [A, R, \beta B, V]$.

Then there exists $[A, R, \beta B, V] \Rightarrow^*_{lm} y'$ in $\boldsymbol{T}(G)$. By applying Lemma 6.2 to this derivation, we have $Ar' \Rightarrow^*_{A,R} \beta B y' r'$ in G, and if $\alpha = \beta\gamma$, then we know that there exists

$$Ar \Rightarrow^*_{A,R} \beta B y' r \Rightarrow_{A,R} \beta\gamma y' r \text{ in } G. \tag{6.3}$$

At this point, we have an LR(k) conflict with the derivations (6.2) and (6.3) because of $k{:}ay_1 r = k{:}y'r'$.

(Subcase 1-2) $p_T^2 = [A, R, A, U] \to \epsilon$.

In this case, $\alpha = A$ and $y' = \epsilon$. It implies that $k{:}yr$ is also in R because of $k{:}yr = k{:}y'r' = r'$ and $r' \in R$. However, $k{:}yr (= k{:}ay_1r)$ cannot be in R by the definition of $\Rightarrow_{A,R}$ because we already have the derivation (6.2).

(Subcase 1-3) $p_T^2 = [A, R, \alpha, U] \to [B, W, \gamma, V][A, R, \beta B, W]$, where $\beta\gamma = \alpha$.

Let $[B, W, \gamma, V] \Rightarrow_{lm}^* y_1'$ and $[A, R, \beta B, W] \Rightarrow_{lm}^* y_2'$ in $\boldsymbol{T}(G)$, where $y_1'y_2' = y'$. By applying Lemma 6.2 to these derivations, there exist $Bw \Rightarrow_{B,W}^* \gamma y_1'w$, where $w \in W$ and $k{:}y_1'w \in V$, and $Ar' \Rightarrow_{A,R}^* \beta B y_2' r'$, where $k{:}y_2'r' = w$ in G. Then we know that $\boldsymbol{\Phi}(A, R, \alpha, k{:}y_1'w) = (B, W, \gamma)$. This is a contradiction to $k{:}ay_1r = k{:}y_1'y_2'r'$ because $k{:}ay_1r$ cannot be in $\{u \in U \mid \boldsymbol{\Phi}(A, R, \alpha, u) \text{ is defined}\}$ according to the construction of $\boldsymbol{T}(G)$.

Case 2. $p_T^1 = [A, R, \alpha, U] \to [A, R, \beta B, V]$.

Then there exists $[A, R, \beta B, V] \Rightarrow_{lm}^* y$ in $\boldsymbol{T}(G)$. By applying Lemma 6.2 to this derivation, we have $Ar \Rightarrow_{A,R}^* \beta Byr$ in G. If $\alpha = \beta\gamma$, then $B \to \gamma \in P$ and there exists

$$Ar \Rightarrow_{A,R}^* \beta Byr \Rightarrow_{A,R} \beta\gamma yr \text{ in } G. \tag{6.4}$$

(Subcase 2-1) $p_T^2 = [A, R, \alpha, U] \to [A, R, \zeta C, Z]$.

Then there exists $[A, R, \zeta C, Z] \Rightarrow_{lm}^* y'$ in $\boldsymbol{T}(G)$. By applying Lemma 6.2 to this derivation, there exists $Ar' \Rightarrow_{A,R}^* \zeta C y' r'$ in G, and if $\alpha = \zeta\delta$, then there exists

$$Ar' \Rightarrow_{A,R}^* \zeta C y' r' \Rightarrow_{A,R} \zeta\delta y' r' \text{ in } G. \tag{6.5}$$

At this time, we have an LR(k) conflict with the derivations (6.4) and (6.5) because of $B \to \gamma \neq C \to \delta$, although $\beta\gamma = \zeta\delta$ and $k{:}yr = k{:}y'r'$.

(Subcase 2-2) $p_T^2 = [A, R, A, U] \to \epsilon$.

In this case, $\alpha = A$ and $y' = \epsilon$. If $\beta = \epsilon$, then $\gamma = A$, and we have a contradiction to derivation (6.4) similar to (Subcase 1-2). If $\beta \neq \epsilon$, then $\beta = A$ and $\gamma = \epsilon$. Then there exists $Ar \Rightarrow_{A,R}^* AByr \Rightarrow_{A,R} Ayr$ in G. Here, $k{:}yr = k{:}y'r'$, and so $k{:}yr \in R$ because of $y' = \epsilon$ and $r' \in R$. However, it is a contradiction to the definition of $\Rightarrow_{A,R}$.

(Subcase 2-3) $p_T^2 = [A, R, \alpha, U] \to [C, X, \delta, Z][A, R, \zeta C, X]$, where $\zeta\delta = \alpha$.

In this case, we have a contradiction for the construction of $\boldsymbol{T}(G)$ in a similar way to (Subcase 1-3).

Case 3. $p_T^1 = [A, R, A, U] \to \epsilon$.

Then $\alpha = A$, $y = \epsilon$, and the only possible form of p_T^2 is $[A, R, \alpha, U] \to [B, W, \gamma, V]$ $[A, R, \beta B, W]$, where $\beta\gamma = \alpha$. Let $[B, W, \gamma, V] \Rightarrow_{lm}^* y_1'$ and $[A, R, \beta B, W] \Rightarrow_{lm}^* y_2'$ in $\boldsymbol{T}(G)$, where $y_1'y_2' = y'$. By applying Lemma 6.2 to these derivations, we have $Bw \Rightarrow_{B,W}^* \gamma y_1' w$, where $w \in W$, and $Ar' \Rightarrow_{A,R}^* \beta B y_2' r'$, where $k{:}y_2'r' = w$ in G. Hence, there exists $Ar' \Rightarrow_{A,R}^* \beta B y_2' r' \Rightarrow_{A,R}^* \beta\gamma y_1' y_2' r'$. Note that $\beta\gamma = A$ and $k{:}y_1'y_2'r' = r$ because of $k{:}y_1'y_2'r' = k{:}yr$ and $y = \epsilon$. It means that $k{:}y_1'y_2'r'$ is in R, but this containment cannot occur by the definition of $\Rightarrow_{A,R}$.

Case 4. $p_T^1 = [A, R, \alpha, U] \to [B, W, \gamma, V][A, R, \beta B, W]$, where $\beta\gamma = \alpha$.

Let $[B, W, \gamma, V] \Rightarrow_{lm}^* y_1$ and $[A, R, \beta B, W] \Rightarrow_{lm}^* y_2$ in $\boldsymbol{T}(G)$, where $y_1y_2 = y$. By applying Lemma 6.2 to these derivations, we have $Bw \Rightarrow_{B,W}^* \gamma y_1 w$, where $w \in W$ and $Ar \Rightarrow_{A,R}^* \beta B y_2 r$, where $k{:}y_2r = w$ in G. We note that $\boldsymbol{\Phi}(A, R, \alpha, k{:}y_1w) = (B, W, \gamma)$. On the other hand, the only possible form of p_T^2 is $[A, R, \alpha, U] \to [C, X, \delta, Z][A, R, \zeta C, X]$, where $\zeta\delta = \alpha$. Let $[C, X, \delta, Z] \Rightarrow_{lm}^* y_1'$ and $[A, R, \zeta C, X] \Rightarrow_{lm}^* y_2'$ in $\boldsymbol{T}(G)$, where $y_1'y_2' = y'$. By applying Lemma 6.2 to these derivations, we have $Cx' \Rightarrow_{C,X}^* \delta y_1' x'$, where $x' \in X$ and $k{:}y_1'x' \in Z$, and $Ar' \Rightarrow_{A,R}^* \zeta C y_2' r'$, where $k{:}y_2'r' = x'$ in G. Here,

$\mathbf{\Phi}(A, R, \alpha, k{:}y_1'x') = (C, X, \delta)$. Then we have a contradiction to the definition of $\mathbf{\Phi}$ function because of $k{:}y_1 w = k{:}y_1'x'$.

For all the possible cases of p_T^1 and p_T^2, we showed some contradictions, and hence $\boldsymbol{T}(G)$ has to be LL(k). □

From Theorems 6.5 and 6.6, we obtain that $\boldsymbol{T}(G)$ is an LL(k) covering grammar of G.

6.3. Transformable grammars. We define some special nonterminals to detect the infinite process of Algorithm 1. Let $[A, R, \alpha, U] \in \boldsymbol{N}$ and $\alpha = X_1 \cdots X_n$. Assume that $q_i = \{[B \to \gamma.\delta, k{:}zr] |$ there exists $Ar \Rightarrow^*_{A,R} \beta B zr \Rightarrow_{A,R} \beta\gamma\delta zr \Rightarrow^*_{A,R} \beta\gamma\zeta yzr$ in G, where $r \in R$, $k{:}yzr \in U$, $\beta\gamma = X_1 \cdots X_i$, and $\zeta = X_{i+1} \cdots X_n$ $\}$ for each $i = 0, 1, \ldots, n$. For $q_i, q_{i+1}, \ldots, q_j$, if $q_i = q_j (0 \le i < j \le n)$ and no other pair of $q_i, q_{i+1}, \ldots, q_j$ is identical, then $q_i, \ldots, q_j$ is a *loop*. We say that $[A, R, \alpha, U]$ is *cyclic* if there exist more than two different values for i, $0 \le i \le n$ for the same loop such that $q_i, \ldots, q_j$ is a loop. If $[A, R, \alpha, U]$ is cyclic and there is no $u \in U$ such that $\Phi(A, R, \alpha, u) = (B, W, \gamma)$ for some B, W, and γ, then $[A, R, \alpha, U]$ is an *indivisible cyclic*. Using this nonterminal, we can characterize the transformable grammars as follows.

LEMMA 6.7. *Algorithm* 1 *with G successfully terminates iff no indivisible cyclic nonterminal is generated.*

Proof. (*Only if part*) Assume that an indivisible cyclic nonterminal $[A, R, \alpha, U]$ is generated during the execution of Algorithm 1 with G. Then an infinite number of cyclic nonterminals that have the common loop with $[A, R, \alpha, U]$ are generated according to Algorithm 1. Hence, Algorithm 1 does not successfully terminate.

(*If part*) During the execution of Algorithm 1, if no indivisible cyclic nonterminal is generated, then the length of each nonterminal generated is bounded. It means that Algorithm 1 successfully terminates. □

As a result, whenever an indivisible cyclic nonterminal is found during the working of Algorithm 1, it is desirable to stop anymore processing of the algorithm.

Next we will give another characterization of the transformable grammars using grammatical derivations.

THEOREM 6.8. *Algorithm* 1 *with G successfully terminates iff (Statement* 1*) is true:*

(*Statement* 1) *There exists a constant n, depending on G, such that if $\alpha (\ne \epsilon)$ is a viable prefix of G, x is a string in $L^G(\alpha)$, and v is a string in $RC_k^G(\alpha)$, then there exist B, W, and $\gamma (\ne \epsilon)$, where $\alpha = \beta\gamma$ and $|\gamma| \le n$ such that Condition 1 holds.*

(*Condition* 1) *Whenever there exists π such that $S' \Rightarrow^{\pi}_{rm} \beta\gamma z \Rightarrow^*_{rm} xz$ in G, where $k{:}z = v$, there exist π^1 and π^2, where $\pi^1\pi^2 = \pi$ such that $S' \Rightarrow^{\pi^1}_{rm} \beta B z''$ and $Bw \Rightarrow^{\pi^2}_{B,W} \gamma z' w$ in G, where $w = k{:}z''$ $(w \in W)$ and $z'z'' = z$.*

Proof. *(Only if part)* Assume that Algorithm 1 with G successfully terminates, but there is no n satisfying Statement 1. In other words, there exist a viable prefix α of G, a string x, $x \in L^G(\alpha)$, and a string v, $v \in RC_k^G(\alpha)$ for which there are no predicted B, W, and γ satisfying Condition 1, where $\alpha = \beta\gamma$ and $|\gamma| \le n$ for some fixed n. Take an arbitrary rule string $\hat{\pi} (\hat{\pi} = \pi'\pi'')$ such that there exists $S\$^k \Rightarrow^{\pi'}_{rm} \alpha z \Rightarrow^{\pi''}_{rm} xz$ in G, where $k{:}z = v$. Corresponding to $\hat{\pi} (= \pi'\pi'')$, there exists $\hat{\pi}_T$ such that $[S, \{\$^k\}, \epsilon, FIRST_k(S\$^k)] \Rightarrow^{\hat{\pi}_T}_{lm} xz$ in $\boldsymbol{T}(G)$, where $h(\hat{\pi}_T) = \hat{\pi}^R$ according to Lemma 6.1. Furthermore, we know that $\hat{\pi}_T$ is composed of π_T'' and π_T' such that $[S, \{\$^k\}, \epsilon, FIRST_k(S\$^k)] \Rightarrow^{\pi_T''}_{lm} \bar{x}\tau \Rightarrow^{\pi_T'}_{lm} \bar{x}\bar{z}$ in $\boldsymbol{T}(G)$, $\tau \in \boldsymbol{N}^*$, where $\bar{x}\bar{z} = xz$, $h(\pi_T'')^R = \pi''$, and $h(\pi_T')^R = \pi'$.

Let τ be $[B_n, W_n, \gamma_n, U_n][B_{n-1}, W_{n-1}, \gamma_{n-1}B_n, W_n] \cdots [B_1, W_1, \gamma_1 B_2, W_2]$. Suppose that $\pi_T^i = [B_i, W_i, \gamma_i B_{i+1}, W_{i+1}] \Rightarrow_{lm}^* z_i$ for each $i = 1, \ldots, n-1$ and $\pi_T^n = [B_n, W_n, \gamma_n, U_n] \Rightarrow_{lm}^* z_n$, where $\pi_T^1 \cdots \pi_T^n = \pi_T'$ and $z_n \cdots z_1 = \bar{z}$. Then by applying Lemma 6.2 to $\pi_T^1, \ldots, \pi_T^n$, respectively, we have $B_i w_i \Rightarrow_{B_i, W_i}^{h(\pi_T^i)^R} \gamma_i B_{i+1} z_i w_i$, where $w_i \in W_i$ and $k{:}z_i w_i \in W_{i+1}$ for each $i = 1, \ldots, n-1$, and $B_n w_n \Rightarrow_{B_n, W_n}^{h(\pi_T^n)^R} \gamma_n z_n w_n$, where $w_n \in W_n$ and $k{:}z_n w_n \in U_n$ in G. Because of $\pi_T^n \pi_T^{n-1} \cdots \pi_T^1 = \pi_T'$, $h(\pi_T^1)^R \cdots h(\pi_T^n)^R = (h(\pi_T^n) \cdots h(\pi_T^1))^R = h(\pi_T^n \cdots \pi_T^1)^R = h(\pi_T')^R = \pi'$. Here, we have $z = \bar{z}$, and so $x = \bar{x}$; $\gamma_1 \cdots \gamma_n = \alpha$.

On the other hand, we know that π_T'' is of the form

$$
\begin{aligned}
&[S, \{\$^k\}, \epsilon, FIRST_k(S\$^k)] \\
&\Rightarrow_{lm}^{\pi_T^X} x'[B_{n-1}, W_{n-1}, \gamma_{n-1}\delta_n, X_{n-1}] \cdots [B_1, W_1, \gamma_1 B_1, W_2] \\
&\Rightarrow_{lm}^{r_T} x'[B_n, W_n, \delta_n, V_n][B_{n-1}, W_{n-1}, \gamma_{n-1}B_n, W_n] \cdots [B_1, W_1, \gamma_1 B_1, W_2] \\
&\Rightarrow_{lm}^{\pi_T^Y} x'x''[B_n, W_n, \gamma_n, U_n][B_{n-1}, W_{n-1}, \gamma_{n-1}B_n, W_n] \cdots [B_1, W_1, \gamma_1 B_1, W_2],
\end{aligned}
$$

where $x'x'' = x$. Hence, we have $\Phi(B_{n-1}, W_{n-1}, \gamma_{n-1}\delta_n, u) = (B_n, W_n, \delta_n)$, where $u = k{:}x''z$. Consequently, we have the following statement:

(Statement 2) whenever there exists $B_{n-1}w_{n-1} \Rightarrow_{B_{n-1}, W_{n-1}}^* \gamma_{n-1}\delta_n p w_{n-1}$, where $w_{n-1} \in W_{n-1}$ and $k{:}pw_{n-1} = u$, there exist $B_{n-1}w_{n-1} \Rightarrow_{B_{n-1}, W_{n-1}}^* \gamma_{n-1}B_n q w_{n-1}$ and $B_n w_n \Rightarrow_{B_n, W_n}^* \delta_n r w_n$, where $w_n = k{:}qw_{n-1}$ and $rq = p$. Additionally, we know that when α, x, and v are fixed, $B_1, W_1, \gamma_1, \ldots, B_i, W_i, \gamma_i, \ldots, B_n, W_n, \gamma_n, \delta_n, U_n$ are also fixed because $\boldsymbol{T}(G)$ is LL(k).

Consider the derivation $\pi_T^s = r_T \pi_T^Y \pi_T^n \pi_T^{n-1}$ such that $[B_{n-1}, W_{n-1}, \gamma_{n-1}\delta_n, X_{n-1}] \Rightarrow_{lm}^{\pi_T^s} x'' z_n z_{n-1}$. Then by applying Lemma 6.2, there exists $h(\pi_T^s)^R$ such that $B_{n-1}w_{n-1} \Rightarrow_{B_{n-1}, W_{n-1}}^* \gamma_{n-1}\delta_n x'' z_n z_{n-1} w_{n-1}$ in G, where $w_{n-1} \in W_{n-1}$ and $k{:}x''z_n z_{n-1} w_{n-1} \in X_{n-1}$. Here, $h(\pi_T^s)^R = h(r_T \pi_T^Y \pi_T^n \pi_T^{n-1})^R = h(\pi_T^Y \pi_T^n \pi_T^{n-1})^R = h(\pi_T^{n-1})^R h(\pi_T^n)^R h(\pi_T^Y)^R$ because of $h(r_T) = \epsilon$. Since π_T^Y is a suffix of π_T'', $h(\pi_T^Y)^R$ is a prefix of π''. Let π^Z be the rule string such that $\pi'' = h(\pi_T^Y)^R \pi^Z$. Then $\hat{\pi}$ is composed of the derivations such that

$$
\begin{aligned}
&B_1 w_1 \\
&\Rightarrow_{rm}^{h(\pi_T^1)^R h(\pi_T^2)^R \cdots h(\pi_T^{n-2})^R} \gamma_1\gamma_2\gamma_3 \cdots \gamma_{n-2} B_{n-1} z_{n-2} \cdots z_2 z_1 w_1 \\
&\Rightarrow_{rm}^{h(\pi_T^{n-1})^R} \gamma_1\gamma_2\gamma_3 \cdots \gamma_{n-1} B_n z_{n-1} \cdots z_2 z_1 w_1 \\
&\Rightarrow_{rm}^{h(\pi_T^n)^R} \gamma_1\gamma_2 \cdots \gamma_{n-1}\gamma_n z_n z_{n-1} \cdots z_2 z_1 w_1 \\
&\Rightarrow_{rm}^{h(\pi_T^Y)^R} \gamma_1\gamma_2 \cdots \gamma_{n-1}\delta_n x'' z_n z_{n-1} \cdots z_2 z_1 w_1 \\
&\Rightarrow_{rm}^{\pi^Z} x'x'' z_n z_{n-1} \cdots z_2 z_1 w_1,
\end{aligned}
$$

where $B_1 = S$, $w_1 = \$^k$, $\gamma_1 \cdots \gamma_n = \alpha$, and $z_n z_{n-1} \cdots z_2 z_1 = z$. Then we have that whenever there exists π such that $S' \Rightarrow_{rm}^{\pi} \alpha z \Rightarrow_{rm}^* xz$ in G, where $k{:}z = v$, there exists $\tilde{\pi}(= \pi\pi'')$ such that $S' \Rightarrow_{rm}^{\pi} \alpha z \Rightarrow_{rm}^{\pi''} \gamma_1 \cdots \gamma_{n-1}\delta_n yz \Rightarrow_{rm}^* xz$ in G. The derivation $\tilde{\pi}$ is always composed of $\tilde{\pi}_1$ and $\tilde{\pi}_2$ such that $\tilde{\pi}_1 = S' \Rightarrow_{rm}^* \gamma_1 \cdots \gamma_{n-1} B_{n-1} z_2$ and $\tilde{\pi}_2 = B_{n-1}w_{n-1} \Rightarrow_{B_{n-1}, W_{n-1}}^* \gamma_n z_1 w_{n-1} \Rightarrow_{B_{n-1}, W_{n-1}}^* \delta_n y z_1 w_{n-1}$, where $w_{n-1} = k{:}z_2$. At this point, if we set $\beta = \gamma_1 \cdots \gamma_{n-1}$, $B = B_n$, $W = W_n$, $\gamma = \gamma_n$, and $U = U_n$, then from Statement 2, we obtain the property that whenever there exists π such that $S' \Rightarrow_{rm}^{\pi} \alpha z \Rightarrow_{rm}^* xz$ in G, where $k{:}z = v$, there exist π^1 and π^2, where $\pi^1\pi^2 = \pi$ such that $S' \Rightarrow^{\pi^1}_{rm} \beta B z''$ and $Bw \Rightarrow^{\pi^2}_{B,W} \gamma z' w$ in G, where $w = k{:}z''$ and $z'z'' = z$. Here, $|\gamma|$ is bounded since it is a component of the nonterminal $[B, W, \gamma, U]$. As a result, it is a contradiction of the assumption that α and v have no predictable B, W, and γ satisfying Condition 1.

(If part) Assume that Statement 1 is true, but Algorithm 1 with G does not successfully terminate. It means that an indivisible cyclic nonterminal is generated, while Algorithm 1 with G works. Let $[A, R, \alpha, U]$ be such a nonterminal. Suppose that n is the constant in Statement 1. Without loss of generality, we can assume $|\alpha| > n$.

Take an arbitrary derivation

$$Ar \Rightarrow^{*}_{A,R} \alpha y r \Rightarrow^{*}_{A,R} \bar{x} y r \tag{6.6}$$

in G, where $r \in R$. Let $v = k{:}yr$. Then there exists $S' \Rightarrow^{*}_{rm} \theta A z \Rightarrow^{*}_{rm} \theta\alpha y z \Rightarrow^{*}_{rm} xyz$ in G, where $k{:}z = r$, $k{:}yz = v$, and $x = \hat{x}\bar{x}$. If we consider the viable prefix $\theta\alpha$, the string $x, x \in L^{G}(\theta\alpha)$, and the string u, $u \in RC^{G}_{k}(\theta\alpha)$, then there exist B, W, and γ, where $\theta\alpha = \beta\gamma$ and $|\gamma| \le n$ satisfying Condition 1 since Statement 1 is true. That is, whenever there exists π such that $S' \Rightarrow^{\pi}_{rm} \beta\gamma z \Rightarrow^{*}_{rm} xyz$ in G, where $k{:}z = v$, there exist π^1 and π^2, where $\pi^1\pi^2 = \pi$ such that $S' \Rightarrow^{\pi^1}_{rm} \beta B z''$ and $Bw \Rightarrow^{\pi^2}_{B,W} \gamma z' w$ in G, where $z'z'' = z$ and $w = k{:}z''$. Hence, we have the property that: whenever there exists a derivation (6.6), there exist $Ar \Rightarrow^{*}_{A,R} \delta B z''' r$ and $Bw \Rightarrow^{*}_{B,W} \gamma z' w$, where $z'z''' = y$ and $\alpha = \delta\gamma$. Consequently, we have the property: whenever there exists $Ar \Rightarrow^{*}_{A,R} \delta\gamma y r$, where $r \in R$ and $k{:}yr = v$, there exist $Ar \Rightarrow^{*}_{A,R} \delta B z''' r$ and $Bw \Rightarrow^{*}_{B,W} \gamma z' w$, where $z'z''' = y$ and $w = k{:}z''' r$. In accordance with Lemma 4.4, $(A, R, \delta\gamma)\mathbf{\Pi}_{v}(B, W', \gamma)$ holds for some W', and hence, there exist B', W'', and γ' such that $\mathbf{\Phi}(A, R, \delta\gamma, v) = (B', W'', \gamma')$ holds. Then we have a contradiction to that $[A, R, \alpha, U]$ is indivisible cyclic. □

In other words, the transformable grammars have a fixed constant n: when the stack string is $\alpha(\neq \epsilon)$, the input string already read is x, and the k-length prefix of the remaining input is v, we always have $B, W, \gamma(\neq \epsilon)$, where $|\gamma| \le n$ such that γ and some prefix of the remaining input will be reduced to (B, W). Figure 6.1 describes this situation; the left side is expected to be the right.

We define the transformable grammars using Algorithm 1 as extended PLR(k) grammars.

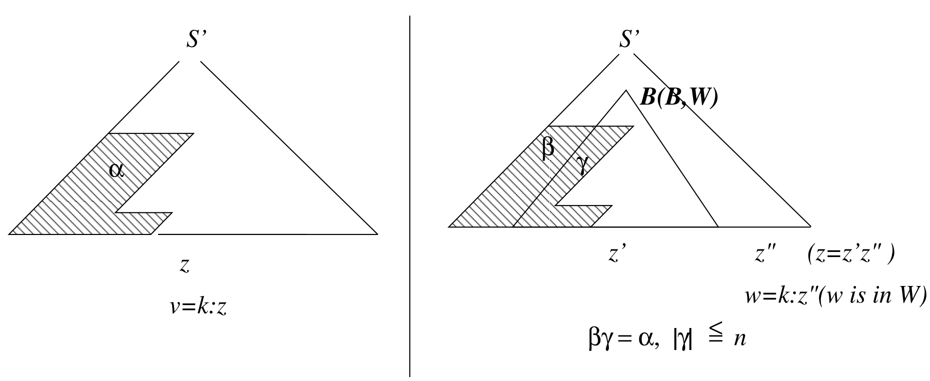


FIG. 6.1. *Derivation trees.*

Definition 6.9. *G is an extended PLR(k) grammar iff Statement 1 is true.*

We next relate extended PLR(k) grammars with PLR(k) grammars and k-transformable grammars.

Lemma 6.10. *Let G be PLR(k). Then G is extended PLR(k).*

Proof. Suppose that $\alpha(\neq \epsilon)$ is a viable prefix of G, x is a string in $L^G(\alpha)$, and v is a string in $RC_k^G(\alpha)$. Then there exists a derivation $S' \Rightarrow_{rm}^* \beta Bz \Rightarrow_{rm} \beta X\delta\zeta z \Rightarrow_{rm}^* \beta X\delta y_2 z \Rightarrow_{rm}^* \beta X y_1 y_2 z \Rightarrow_{rm}^* x y_2 z$ in G, where $\beta X \delta = \alpha$, $k{:}y_2 z = v$, and $X\delta \neq \epsilon$. On the other hand, according to the definition of PLR(k) grammars,

$$\text{if there exists } S' \Rightarrow_{rm}^* \beta' A z' \Rightarrow_{rm} \beta'\beta'' X\delta'\zeta' z' \Rightarrow_{rm}^* \beta'\beta'' X\delta' y_2' z'$$

$$\Rightarrow_{rm}^* \beta'\beta'' X y_1' y_2' z' \text{ in } G, \text{ where } \beta'\beta'' = \beta \text{ and } k{:}y_1' y_2' z' = k{:}y_1 y_2 z,$$

$$\text{then } A = B \text{ and } \beta'' = \epsilon. \tag{6.7}$$

Let n be the maximum length of the right-side string of a rule of G and γ be $X\delta$. Then we know that $|\gamma| \leq n(\gamma \neq \epsilon)$. On the other hand, whenever there exists $S' \Rightarrow_{rm}^* \beta\gamma z \Rightarrow_{rm}^* xz$ in G, there exists $S' \Rightarrow_{rm}^* \beta\gamma z \Rightarrow_{rm}^* \beta X y z \Rightarrow_{rm}^* xz$ in G and, according to (6.7), there exist $S' \Rightarrow_{rm}^* \beta B z_2$ and $Bw \Rightarrow_{rm}^* \gamma z_1 w$ in G, where $w = k{:}z_2$ and $z_1 z_2 = z$. Let us set W to be $FOLLOW_k(B)$. Then we have the property that whenever there exists π such that $S' \Rightarrow_{rm}^{\pi} \beta\gamma z \Rightarrow_{rm}^* xz$ in G, there exist π^1 and π^2 such that $S' \Rightarrow_{rm}^{\pi^1} \beta B z_2$ and $Bw \Rightarrow_{B,W}^{\pi^2} \gamma z_1 w$ in G, where $\pi^1\pi^2 = \pi$, $w = k{:}z_2$, and $z_1 z_2 = z$. In all, G is extended PLR(k). ☐

Lemma 6.11. *Let G be k-transformable. Then G is extended PLR(k).*

Proof. Since G is k-transformable, there exists a constant n such that if $\alpha(\neq \epsilon)$ is a viable prefix of G and v is a string in $RC_k^G(\alpha)$, then there exist B, W, and $\gamma(\neq \epsilon)$, where $\alpha = \beta\gamma$, $|\gamma| \leq n$, and $v \in RC_k^{B,W}(\gamma)$ such that whenever there exists $S' \Rightarrow_{rm}^* \beta\gamma z \Rightarrow_{rm}^* \beta y z$ in G, where $k{:}yz \in RC_k^{B,W}(\epsilon)$, there exist $S' \Rightarrow_{rm}^* \beta B z''$ and $Bw \Rightarrow_{B,W}^* \gamma z' w \Rightarrow_{B,W}^* y z' w$ in G, where $w = k{:}z''(w \in W)$ and $z'z'' = z$. Note that the condition $k{:}yz \in RC_k^{B,W}(\epsilon)$ implies $k{:}z \in RC_k^{B,W}(\gamma)$. Hence, we have the property with the above n such that if $\alpha(\neq \epsilon)$ is a viable prefix of G, $x \in L^G(\alpha)$, and $v \in RC_k^G(\alpha)$, then there exist B, W, and $\gamma(\neq \epsilon)$, where $\alpha = \beta\gamma$, $|\gamma| \leq n$, and $v \in RC_k^{B,W}(\gamma)$ such that whenever there exists π such that $S' \Rightarrow_{rm}^{\pi} \beta\gamma z \Rightarrow_{rm}^* xz$ in G, where $k{:}z \in RC_k^{B,W}(\gamma)$, there exist π^1 and π^2, where $\pi^1\pi^2 = \pi$ such that $S' \Rightarrow_{rm}^{\pi^1} \beta B z''$ and $Bw \Rightarrow_{B,W}^{\pi^2} \gamma z' w$ in G, where $w = k{:}z''(w \in W)$ and $z'z'' = z$. Thus, G is extended PLR(k). ☐

We showed that extended PLR(k) grammars completely contain PLR(k) grammars and k-transformable grammars. Extended PLR(k) grammars, moreover, are larger than k-transformable grammars and PLR(k) grammars. It can be proved by the following example.

Example 6.1. Let $G3 = (\{S, A\}, \{a, d, b, c\}, \{S \to aAd, S \to aB, B \to aA, A \to ab, A \to aA, A \to bc\}, S)$. Note that $G3$ is LR(1) and extended PLR(1). However, $G3$ is neither k-transformable nor PLR(k) for all $k \geq 1$.

Theorem 6.12. *Extended PLR(k) grammars are larger than PLR(k) grammars and k-transformable grammars.*

7. Concluding remarks. The contribution of this paper can be summarized as follows. First, we showed the incomparable relationship between PLR(k) grammars and k-transformable grammars. Second, we presented the generalization of the reduction goal prediction in LR parsing. We believe that the proposed one is the uppermost of the reduction goal prediction, which is performed by keeping the LR stack and investigating the k-length prefix of the remaining input. Extended PLR(k) grammars are thus thought of as the uppermost class of LL(k) covering transformable grammars based on such a prediction. Third, we showed that LL(k) covering grammars can be deterministically constructed by defining the $\mathbb{\Pi}$ relation and investigating the orderable property of the relation. As a result, we can decide the transformableness of a given grammar in a single process. Lastly, we characterized the transformable grammars using grammatical derivation.

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